



Process Systems Engineering and Process Safety

Output feedback robust model predictive control with unmeasurable model parameters and bounded disturbance☆



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ABSTRACT

The output feedback model predictive control (MPC), for a linear parameter varying (LPV) process system including unmeasurable model parameters and disturbance (all lying in known polytopes), is considered. Some previously developed tools, including the norm-bounding technique for relaxing the disturbance-related constraint handling, the dynamic output feedback law, the notion of quadratic boundedness for specifying the closed-loop stability, and the ellipsoidal state estimation error bound for guaranteeing the recursive feasibility, are merged in the control design. Some previous approaches are shown to be the special cases. An example of continuous stirred tank reactor (CSTR) is given to show the effectiveness of the proposed approaches.

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1. Introduction

Consider a process system model

$$\mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k) + \mathbf{D}(k)\mathbf{w}(k), \quad (1)$$

$$\mathbf{y}(k) = \mathbf{C}(k)\mathbf{x}(k) + \mathbf{E}(k)\mathbf{w}(k),$$

where $\mathbf{u} \in \mathbb{R}^{n_u}$, $\mathbf{x} \in \mathbb{R}^{n_x}$, $\mathbf{y} \in \mathbb{R}^{n_y}$ and $\mathbf{w} \in \mathbb{R}^{n_w}$ are the input, state, output and bounded disturbance, respectively. This model has been widely studied (see e.g. [1]). For the time-varying $[\mathbf{A}|\mathbf{B}|\mathbf{C}|\mathbf{D}|\mathbf{E}](k)$ lying in a polytope, two kinds of uncertainties have been considered. One, which is referred to as the quasi-uncertain linear parameter varying (LPV), assumes that $[\mathbf{A}|\mathbf{B}|\mathbf{C}|\mathbf{D}|\mathbf{E}](k)$ are exactly known at the current time k albeit unknown in the future. The other, which is referred to as the uncertain LPV, assumes that $[\mathbf{A}|\mathbf{B}|\mathbf{C}|\mathbf{D}|\mathbf{E}](k)$ are unknown for all k .

For the case when $\mathbf{w}(k)$ is persistent and unknown, we have published several works for output feedback robust MPC (OFRMPC) (see Table 1). In these previous works, we have developed the following effective technical tools:

- (1) norm-bounding technique for handling physical constraints. Some scalars (norm bounds) are utilized to counteract the

effect of $\mathbf{w}(k)$. These scalars are off-line optimized, so that the conservatism is reduced for constraint handling.

- (2) the notion of quadratic boundedness (QB) and dynamic output feedback law. QB describes quadratic convergence outside of an ellipsoid and invariance inside of this ellipsoid, which is concise for specifying stability property. The dynamic output feedback has four control parametric matrices, which can admit desirable solutions for the uncertain LPV models (the computational burden can be reduced by pre-fixing two control parametric matrices), and parameter-dependent efficient solutions for the quasi-uncertain LPV models.
- (3) the ellipsoidal and polyhedral state estimation error bounds (EEBs). The ellipsoidal and polyhedral sets are widely utilized in the control theory. We utilize both kinds of sets which have complementary advantages.
- (4) the auxiliary optimization problem for determining the time for performing the main optimization, which also reduces the on-line computational burden. An inappropriate refreshment of the main optimization problem may not yield the feasible solutions, and it is unnecessary that receding-horizon optimization always improves the control performance.
- (5) the ellipsoid refreshment formula for guaranteeing the recursive feasibility. By a simple refreshment of the ellipsoidal EEB, the recursive feasibility can be guaranteed.

As shown in Table 1, not every previous work utilizes all the above five tools. Previously, the above tools were not fully utilized due to

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Table 1

The differences between this paper and the existing works

	System model	Disturbance bound	EEC	Tool (1)	QB in tool (2)	EEB of tool (3)	Tool (4)	Tool (5)
[2]	Quasi-uncertain LPV	Ellipsoidal			✓	Polyhedral	✓	
[3]	Quasi-uncertain LPV	Ellipsoidal			✓	Ellipsoidal		✓
[4]	Uncertain LPV	Ellipsoidal			✓	Ellipsoidal		✓
[5]	Hammerstein–Wiener	Polyhedral	✓	✓		Polyhedral		
[6]	Uncertain LPV	Ellipsoidal	✓		✓	Polyhedral		
[7]	Hammerstein–Wiener	Ellipsoidal			✓	Ellipsoidal		✓
[8]	Uncertain LPV	Polyhedral	✓	✓		Polyhedral		
[14]	Uncertain LPV	Ellipsoidal			✓	Polyhedral	✓	
[15]	Quasi-uncertain LPV	Ellipsoidal			✓	Ellipsoidal & polyhedral		✓
This paper	Uncertain LPV	Polyhedral		✓	✓	Ellipsoidal		✓

technical difficulties. In our recent works [16,17], in the context of polyhedral disturbance bound and uncertain LPV model, the above techniques (1), (2) and (3) have been merged. In this paper, we summarize the results in [16] and [17], and give a set of algorithms for OFRMPC; this is mainly related to [4], [6], and [8], for which more comparisons are shown in the following.

The work [6] firstly applies the following dynamic output feedback controller for MPC of the uncertain LPV system:

$$\mathbf{x}_c(k+1) = \mathbf{A}_c(k)\mathbf{x}_c(k) + \mathbf{L}_c(k)\mathbf{y}(k), \quad \mathbf{u}(k) = \mathbf{F}_x(k)\mathbf{x}_c(k) + \mathbf{F}_y(k)\mathbf{y}(k) \quad (2)$$

where $\mathbf{x}_c \in \mathbb{R}^{n_{xc}}$ is the controller state; $\{\mathbf{A}_c, \mathbf{L}_c\}(k)$ are controller gain matrices; $\{\mathbf{F}_x, \mathbf{F}_y\}(k)$ are feedback gain matrices. Indeed, before [6], the works [5,8] apply the same feedback controller, albeit $\mathbf{F}_y = 0$ and originally discussing the estimator-based output feedback. In the context of OFRMPC, one of the major difficulties is refreshing EEBs. The works [5,6,8] add the estimation error constraint (EEC) in the main optimization problem which imposes that the estimation error has to evolve within the polyhedral EEBs. Adding EEC simplifies OFRMPC since it is no longer required to refresh EEB. However, EEC has its adverse effects, such as degrading the control performance and shrinking the region of attraction. In [2] and [14], EEC is removed, and the auxiliary optimization problems are utilized to refresh EEBs. By applying the polyhedral EEBs (see [2,5,6,8,14]), the recursive feasibility of the main optimization problem is not guaranteed. By applying the ellipsoidal EEB (see [3,4,7,15]), it begins to guarantee the recursive feasibility.

Some previous works are not for uncertain LPV models, e.g., [2] and [3] study the quasi-uncertain LPV models and [5,7] consider Hammerstein–Wiener model. The works [5] and [7] are based on those for uncertain LPV models, i.e., [8] and [4], respectively. By removing EEC and optimizing more controller parameters, [4] improves [8], so [7] improves [5]. In the present paper, we study the uncertain LPV model which is more general and sophisticated, so the results can be extended to the quasi-uncertain LPV and Hammerstein–Wiener models.

There are several more differences among [4], [6], and [8]. The work [8] applies the dilation technique to handle the linear matrix inequalities (LMIs), while [4] and [6] abandon this technique. The work [8] uses the norm-bounding technique to handle the invariance conditions, while [4] and [6] utilize the notion of QB. The works [6,8] pre-specify a part of control law parameters for simplification, while [4] optimizes all the controller law parameters. The work [8] assumes polyhedral bound of $\mathbf{w}(k)$ and utilizes the norm-bounding technique to simplify the LMIs, while [4] and [6] assume ellipsoidal bound of $\mathbf{w}(k)$ but give up the norm-bounding technique.

By fusing the merits and erasing the differences, the present paper has three notable contributions: (1) the constraint handling results are improved by the norm-bounding technique; (2) some previous techniques in [4], [6] and [8] are shown to be special cases; (3) it paves the way, so it can be the new starting point, for the future researches.

Notations: For a vector $\mathbf{x}$ and positive-definite matrix $\mathbf{W}$, $\|\mathbf{x}\|_{\mathbf{W}}^2 = \mathbf{x}^T \mathbf{W} \mathbf{x}$, $\varepsilon_{\mathbf{W}} = \{\xi | \xi^T \mathbf{W} \xi \leq 1\}$, and $\mathbf{x}(i|k)$ is the value of $\mathbf{x}(k+i)$ predicting

at time k . $\mathbf{I}$ is the identity matrix with appropriate dimension. $\text{Co}\{\mathbb{S}\}$ denotes a convex hull, i.e., any element belongs to $\text{Co}\{\mathbb{S}\}$ is a combination of elements in the polytope $\mathbb{S}$, with the scalar combining coefficients being nonnegative and summing as 1. The symbol $\star$ induces a symmetric structure in a square matrix. For the MPC decision variables, the time indices are often omitted for simplicity.

2. Problem Statement

Based on the predictions made by Eqs. (1), (2), the augmented prediction model at time k is

$$\tilde{\mathbf{x}}(i+1|k) = \Phi(i, k)\tilde{\mathbf{x}}(i|k) + \Gamma(i, k)\mathbf{w}(k+i), \quad i \geq 0, \quad \tilde{\mathbf{x}}(0|k) = \tilde{\mathbf{x}}(k) \quad (3)$$

where $\tilde{\mathbf{x}} = [\mathbf{x}^T, \mathbf{x}_c^T]^T$, $\Phi(i, k) = \begin{bmatrix} \mathbf{A}(k+i) + \mathbf{B}(k+i)\mathbf{F}_y(k)\mathbf{C}(k+i) & \mathbf{B}(k+i)\mathbf{F}_x(k) \\ \mathbf{L}_c(k)\mathbf{C}(k+i) & \mathbf{A}_c(k) \end{bmatrix}$ and $\Gamma(i, k) = \begin{bmatrix} \mathbf{B}(k+i)\mathbf{F}_y(k)\mathbf{E}(k+i) + \mathbf{D}(k+i) \\ \mathbf{L}_c(k)\mathbf{E}(k+i) \end{bmatrix}$. Let us define $\mathbf{z}(k) = \bar{\mathbf{C}}(k)\mathbf{x}(k) + \bar{\mathbf{E}}(k)\mathbf{w}(k) \in \mathbb{R}^{n_z}$, the constrained signal (e.g., $\mathbf{z} = \mathbf{x}$, $\mathbf{z} = \mathbf{y}$), and $\mathbf{z}'(k) = \bar{\mathbf{C}}(k)\tilde{\mathbf{x}}(k) + \bar{\mathbf{E}}(k)\mathbf{w}(k) \in \mathbb{R}^{n_{z'}}$, the penalized signal (e.g., $\mathbf{z}' = \tilde{\mathbf{x}}$, $\mathbf{z}' = \mathbf{z}$). $\{\mathbf{D}, \mathbf{E}, \bar{\mathbf{E}}, \bar{\mathbf{E}}\}(k)$ are the shaping matrices for $\mathbf{w}(k)$.

We will utilize the following disturbance-free prediction model:

$$\tilde{\mathbf{x}}_u(i+1|k) = \Phi(i, k)\tilde{\mathbf{x}}_u(i|k), \quad i \geq 0, \quad \tilde{\mathbf{x}}_u(0|k) = \tilde{\mathbf{x}}(k).$$

Accordingly, $\mathbf{u}_u(i|k) = \mathbf{F}_x(k)\mathbf{x}_{c,u}(i|k) + \mathbf{F}_y(k)\mathbf{y}_u(i|k)$ is the disturbance-free input prediction, and $\mathbf{z}'_u(i|k) = \bar{\mathbf{C}}(k+i)\tilde{\mathbf{x}}_u(i|k)$ is the disturbance-free penalized signal prediction.

The controllers strictly satisfy the following physical constraints:

$$-\bar{\mathbf{u}} \leq \mathbf{u}(k) \leq \bar{\mathbf{u}}, \quad (4)$$

$$-\bar{\Psi} \leq \Psi \mathbf{z}(k+1) \leq \bar{\Psi}, \quad (5)$$

where $\bar{\mathbf{u}} = [\bar{u}_1, \bar{u}_2, \dots, \bar{u}_{n_u}]^T$; $\bar{\Psi} = [\bar{\psi}_1, \bar{\psi}_2, \dots, \bar{\psi}_q]^T$; $\bar{u}_j > 0$, $j = 1, \dots, n_u$; $\bar{\psi}_j > 0$, $j = 1, \dots, q$; $\Psi \in \mathbb{R}^{q \times n_z}$.

2.1. System assumptions and notion of quadratic boundedness

The following is a general assumption of this paper.

Assumption 1. $[\mathbf{A}|\mathbf{B}|\mathbf{C}|\bar{\mathbf{C}}|\bar{\mathbf{C}}|\bar{\mathbf{D}}\mathbf{w}|\mathbf{E}\mathbf{w}|\bar{\mathbf{E}}\mathbf{w}|\bar{\mathbf{E}}\mathbf{w}](k) \in \Omega := \text{Co}\{[\mathbf{A}_l|\mathbf{B}_l|\mathbf{C}_l|\bar{\mathbf{C}}_l|\bar{\mathbf{C}}_l|\bar{\mathbf{D}}_l\mathbf{w}_l|\mathbf{E}_l\mathbf{w}_l|\bar{\mathbf{E}}_l\mathbf{w}_l|\bar{\mathbf{E}}_l\mathbf{w}_l] | l = 1, \dots, L\}$ where $[\bar{\mathbf{D}}_l\mathbf{w}_l|\mathbf{E}_l\mathbf{w}_l|\bar{\mathbf{E}}_l\mathbf{w}_l|\bar{\mathbf{E}}_l\mathbf{w}_l]$ is a constant matrix, i.e., there exist nonnegative coefficients $\lambda_l(k)$, $l = 1, \dots, L$ such that $\sum_{l=1}^L \lambda_l(k) = 1$ and $[\mathbf{A}|\mathbf{B}|\mathbf{C}|\bar{\mathbf{C}}|\bar{\mathbf{C}}|\bar{\mathbf{D}}\mathbf{w}|\mathbf{E}\mathbf{w}|\bar{\mathbf{E}}\mathbf{w}|\bar{\mathbf{E}}\mathbf{w}](k) = \sum_{l=1}^L \lambda_l(k)[\mathbf{A}_l|\mathbf{B}_l|\mathbf{C}_l|\bar{\mathbf{C}}_l|\bar{\mathbf{C}}_l|\bar{\mathbf{D}}_l\mathbf{w}_l|\mathbf{E}_l\mathbf{w}_l|\bar{\mathbf{E}}_l\mathbf{w}_l|\bar{\mathbf{E}}_l\mathbf{w}_l]$.

The following assumption (1) is equivalent to, while (2) is a special case of, Assumption 1:

- (1) $[A|B|C|D|E|\bar{C}|\bar{C}|\bar{E}](k)$ and $w(k)$ lie in different polytopes;
- (2) $[A|B|C|\bar{C}|\bar{C}](k)$, $[D|E|\bar{E}|\bar{E}](k)$ and $w(k)$ lie in different polytopes.

Remark 1. As the special cases for $[Dw|Ew|\bar{E}w|\bar{E}w](k)$ in Assumption 1, the following two assumptions are equivalent:

- A1 $[Dw|Ew|\bar{E}w|\bar{E}w](k) \in \text{Co}\{[Dw\bar{w}_1|Ew\bar{w}_1|\bar{E}w\bar{w}_1|\bar{E}w\bar{w}_1], -[Dw\bar{w}_1|Ew\bar{w}_1|\bar{E}w\bar{w}_1|\bar{E}w\bar{w}_1], [Dw\bar{w}_2|Ew\bar{w}_2|\bar{E}w\bar{w}_2|\bar{E}w\bar{w}_2], -[Dw\bar{w}_2|Ew\bar{w}_2|\bar{E}w\bar{w}_2|\bar{E}w\bar{w}_2], \dots, [Dw\bar{w}_L|Ew\bar{w}_L|\bar{E}w\bar{w}_L|\bar{E}w\bar{w}_L], -[Dw\bar{w}_L|Ew\bar{w}_L|\bar{E}w\bar{w}_L|\bar{E}w\bar{w}_L]\}$;
- A2 $[Dw|Ew|\bar{E}w|\bar{E}w](k) \in \text{Co}\{[Dw\bar{w}_1|Ew\bar{w}_1|\bar{E}w\bar{w}_1|\bar{E}w\bar{w}_1]\sigma(k), [Dw\bar{w}_2|Ew\bar{w}_2|\bar{E}w\bar{w}_2|\bar{E}w\bar{w}_2]\sigma(k), \dots, [Dw\bar{w}_L|Ew\bar{w}_L|\bar{E}w\bar{w}_L|\bar{E}w\bar{w}_L]\sigma(k)\}$, and $\sigma(k) \in \text{Co}\{-1, 1\}$, which can be seen as the ellipsoidal disturbance bound.

The results in this paper are based on Assumption 1, but are more tightly suitable for the above Assumptions A1–A2.

The notion of QB, originally inherited from [9] and [10], can be utilized on Eq. (3).

Definition 1. The system (3) is quadratically bounded with a common Lyapunov matrix Q_0^{-1} , if, for all allowable $\lambda_j(k+i)$ and $w(k+i)$, $i \geq 0$,

$$\{\tilde{x}(i|k)^T Q_0^{-1} \tilde{x}(i|k) \geq 1 \Rightarrow \tilde{x}(i+1|k)^T Q_0^{-1} \tilde{x}(i+1|k) \leq \tilde{x}(i|k)^T Q_0^{-1} \tilde{x}(i|k)\}.$$

Lemma 1. Consider the system (3). For all allowable $\lambda_j(k+i)$ and $w(k+i)$, $i \geq 0$, the following are equivalent:

- (1) Eq. (3) is quadratically bounded with a common Lyapunov matrix Q_0^{-1} ;
- (2) The ellipsoid $\varepsilon_{Q_0^{-1}}$ is positively invariant for Eq. (3), i.e., $\tilde{x}(k) \in \varepsilon_{Q_0^{-1}}$ leads to $\tilde{x}(i|k) \in \varepsilon_{Q_0^{-1}}$ for all $i \geq 1$.

2.2. Optimization problem

In the dynamic OFRMPC, let us solve, at each time k ,

$$\min_{\gamma, \beta, Q, A_c, L_c, F_x, F_y} \left(\max_{[A|B|C|\bar{C}|\bar{C}|Dw|Ew|\bar{E}w|\bar{E}w]_{(k+i) \in \Omega}} \gamma \right), \quad (6)$$

$$\text{s.t. } -\bar{u} \leq u(i|k) \leq \bar{u}, \quad -\bar{\psi} \leq \psi(i+1|k) \leq \bar{\psi}, \quad i \geq 0, \quad (7)$$

$$\tilde{x}(k) \in \varepsilon_{Q^{-1}}, \quad (8)$$

$$\|\tilde{x}(i|k)\|_{\beta^{-1}Q^{-1}}^2 \geq 1 \Rightarrow \|\tilde{x}(i|k)\|_{\beta^{-1}Q^{-1}}^2 - \|\tilde{x}(i+1|k)\|_{\beta^{-1}Q^{-1}}^2 \geq 0, \quad 0 < \beta \leq 1, \quad i \geq 0 \quad (9)$$

$$\|\tilde{x}_u(i|k)\|_{Q^{-1}}^2 - \|\tilde{x}_u(i+1|k)\|_{Q^{-1}}^2 \geq 1/\gamma [\|\tilde{z}'_u(i|k)\|_{W_z}^2 + \|\tilde{u}_u(i|k)\|_{W_u}^2], \quad i \geq 0, \quad (10)$$

where $W_z > 0$ and $W_u > 0$ are weighting matrices. A stabilizing controller results in a stable $\Phi(i, k)$, so $\lim_{i \rightarrow \infty} \tilde{z}'_u(i|k) = 0$ and $\lim_{i \rightarrow \infty} \tilde{u}_u(i|k) = 0$. Hence,

summing Eq. (10) for all i yields $J(k) := \sum_{i=0}^{\infty} [\|\tilde{z}'_u(i|k)\|_{W_z}^2 + \|\tilde{u}_u(i|k)\|_{W_u}^2] \leq \gamma(k) \|\tilde{x}(k)\|_{Q(k)^{-1}}^2$. According to Eq. (8), $J(k)$ is upper-bounded by $\gamma(k)$. Comparing with Definition 1 and Lemma 1, in Eq. (9) we let $Q_0 = \beta Q$.

Let us define $Q = \begin{bmatrix} Q_1 & Q_2^T \\ Q_2 & Q_3 \end{bmatrix}$ and $M = Q^{-1} = \begin{bmatrix} M_1 & M_2^T \\ M_2 & M_3 \end{bmatrix}$ (see [2,3,6,7,15]). In handling Eq. (8), one should eliminate the unmeasurable $x(k)$. By defining an error signal $e(k) = x(k) + M_1(k)^{-1} M_2(k)^T x_c(k)$, the item $\tilde{x}(k)^T Q^{-1} \tilde{x}(k)$ can be divided into a nominal part and an uncertain part. Let us further choose $M_2 = -E_0 M_1$. Then, it is easy to show that $Q_2 = Q_3 E_0$ and

$$Q = \begin{bmatrix} Q_1 & E_0^T Q_3 \\ Q_3 E_0 & Q_3 \end{bmatrix}, M = \begin{bmatrix} M_1 & -M_1 E_0^T \\ -E_0 M_1 & M_3 \end{bmatrix}. \quad (11)$$

Assumption 2. There exists $M_e(0) > 0$ such that $e(0) = x(0) - E_0^T x_c(0) \in \varepsilon_{M_e(0)}$.

Lemma 2. Let Assumption 2 hold and, at each time $k > 0$, find $M_e(k) > 0$ such that $e(k) \in \varepsilon_{M_e(k)}$. Then, condition (8) is guaranteed if either of the following is satisfied:

- (1) there exist a scalar ρ and symmetric matrices $\{M_1, M_3\}$ such that

$$M_1 \leq \rho M_e(k), \quad (12)$$

$$x_c(k)^T (M_3 - E_0 M_1 E_0^T) x_c(k) \leq 1 - \rho; \quad (13)$$

- (2) there exist a scalar ρ and symmetric matrices $\{Q_1, Q_3\}$ such that

$$\begin{bmatrix} Q_1 - E_0^T Q_3 E_0 & \star \\ I & \rho M_e(k) \end{bmatrix} \geq 0, \quad (14)$$

$$\begin{bmatrix} 1 - \rho & \star \\ x_c(k) & Q_3 \end{bmatrix} \geq 0. \quad (15)$$

The above (1) and (2) are equivalent.

Proof. Applying the definitions of M and e yields

$$\begin{aligned} \tilde{x}(k)^T Q^{-1} \tilde{x}(k) &= \tilde{x}(k)^T M \tilde{x}(k) \\ &= e(k)^T M_1 e(k) + x_c(k)^T (M_3 - M_2 M_1^{-1} M_2^T) x_c(k) \\ &= e(k)^T M_1 e(k) + x_c(k)^T (M_3 - E_0 M_1 E_0^T) x_c(k) \end{aligned}$$

If Eq. (12) holds, then Eq. (13) guarantees Eq. (8). Applying the Schur complement, it is shown that Eqs. (12), (13) are equivalent to Eqs. (14), (15).

3. Synthesis Approaches of Dynamic OFRMPC

We firstly propose a general approach to the dynamic OFRMPC, incorporating the relaxation scalars, in Section 3.1, and then simplify it in Section 3.2 by pre-fixing a part of the control parameters. In Section 3.3, we off-line optimize the relaxation scalars finishing the norm-bounding technique.

3.1. A general synthesis approach of dynamic OFRMPC

A specific synthesis approach of OFRMPC is dependent on how to handle Eqs. (7), (9) and (10).

Lemma 3. Condition (9) for all $\tilde{\mathbf{x}}(i|k)$, and Eq. (10) for all $\tilde{\mathbf{x}}_u(i|k)$, in the sense for all admissible $\mathbf{w}(k+i)$, are equivalent to

$$\sum_{l=1}^L \lambda_l(k+i) \sum_{j=1}^L \lambda_j(k+i) \begin{bmatrix} (1-\alpha_{lj})\mathbf{M} & \star & \star \\ 0 & \beta\alpha_{lj} & \star \\ \Phi_{lj} & \Gamma_l \bar{\mathbf{w}}_j & \mathbf{Q} \end{bmatrix} \geq 0, \quad 0 < \beta \leq 1, \quad (16)$$

$$\sum_{l=1}^L \lambda_l(k+i) \sum_{j=1}^L \lambda_j(k+i) \begin{bmatrix} \mathbf{M} & \star & \star & \star \\ \Phi_{lj} & \mathbf{Q} & \star & \star \\ \mathbf{W}_z^{1/2} \bar{\mathbf{C}}_j & 0 & \gamma \mathbf{I} & \star \\ \mathbf{W}_u^{1/2} [\mathbf{F}_y \mathbf{C}_j \quad \mathbf{F}_x] & 0 & 0 & \gamma \mathbf{I} \end{bmatrix} \geq 0, \quad (17)$$

where $\Phi_{lj} = \begin{bmatrix} \mathbf{A}_l + \mathbf{B}_l \mathbf{F}_y \mathbf{C}_j & \mathbf{B}_l \mathbf{F}_x \\ \mathbf{L}_c \mathbf{C}_j & \mathbf{A}_c \end{bmatrix}$ and $\Gamma_l = \begin{bmatrix} \mathbf{D}_w + \mathbf{B}_l \mathbf{F}_y \mathbf{E}_w \\ \mathbf{L}_c \mathbf{E}_w \end{bmatrix}$, and α_{ij} ($\alpha_{ij} = \alpha_{ji}$) are scalars.

Proof. Refer to “Lemma 3” in [16].

Lemma 3 keeps unchanged if we replace $\mathbf{w}(k)$ by $\mathbf{w}(k)\sigma(k)$ where $\sigma(k) \in \text{Co}\{-1, 1\}$ is unknown and persistent. The summations weighted by $\lambda_l(k+i)$ s and $\lambda_j(k+i)$ s, as in Eqs. (16), (17), are named double convex combinations (DbCCs). Refer to [2] and [3] for handling DbCCs, originally to [11] and [12]. Applying Definition 1 and Lemma 1, it is clear that Eqs. (8), (9) result in

$$\tilde{\mathbf{x}}(i|k) \in \varepsilon_{\mathbf{Q}^{-1}}, \quad i \geq 0. \quad (18)$$

Applying Eq. (18) and the following proposition, the physical constraints are handled.

Proposition 1. Suppose $\mathbf{P} > 0$ is a matrix while $\mathbf{a}$ and $\mathbf{b}$ are vectors with appropriate dimensions. Then, for any scalar $\eta \in (0, 1)$,

$$(\mathbf{a} + \mathbf{b})^T \mathbf{P} (\mathbf{a} + \mathbf{b}) \leq \frac{1}{1-\eta} \mathbf{a}^T \mathbf{P} \mathbf{a} + \frac{1}{\eta} \mathbf{b}^T \mathbf{P} \mathbf{b}.$$

Lemma 4. Suppose at time k , there exist scalars $\{\alpha_{lj}, \beta \leq 1, \rho, \eta_{rs}\}$ ($\alpha_{lj} = \alpha_{jl}$), matrices $\{\mathbf{A}_c, \mathbf{L}_c, \mathbf{F}_x, \mathbf{F}_y\}$ and symmetric matrices $\{\mathbf{M}_1, \mathbf{M}_3, \mathbf{Q}_1, \mathbf{Q}_3\}$ such that Eqs. (11)–(13), (16) and

$$\begin{bmatrix} \mathbf{M} & \star & \star \\ 0 & 1 & \star \\ \frac{1}{\sqrt{1-\eta_{1s}}} \xi_s [\mathbf{F}_y \mathbf{C}_j \quad \mathbf{F}_x] & \frac{1}{\sqrt{\eta_{1s}}} \xi_s \mathbf{F}_y \mathbf{E}_w \bar{\mathbf{w}}_j & \bar{u}_s^2 \end{bmatrix} \geq 0, \quad j = 1, \dots, L, \quad s = 1, \dots, n_u, \quad (19)$$

$$\sum_{l=1}^L \sum_{j=1}^L \lambda_l(k+i) \lambda_j(k+i) \begin{bmatrix} \mathbf{M} & \star & \star \\ 0 & 1 & \star \\ \frac{1}{\sqrt{(1-\eta_{3s})(1-\eta_{4s})}} \Psi_s \bar{\mathbf{C}}_h \Phi_{lj}^1 & \frac{1}{\sqrt{(1-\eta_{3s})\eta_{4s}}} \Psi_s \bar{\mathbf{C}}_h \Gamma_l^1 \bar{\mathbf{w}}_j & \bar{\psi}_s^2 - \frac{1}{\eta_{3s}} (\Psi_s \bar{\mathbf{E}}_w \bar{\mathbf{w}}_h)^2 \end{bmatrix} \geq 0, \quad h = 1, \dots, L, \quad (20)$$

$s = 1, \dots, q$

where $\Phi_{lj}^1 = [\mathbf{A}_l + \mathbf{B}_l \mathbf{F}_y \mathbf{C}_j \quad \mathbf{B}_l \mathbf{F}_x]$, $\Gamma_l^1 = \mathbf{D}_w + \mathbf{B}_l \mathbf{F}_y \mathbf{E}_w$, and Ψ_s (ξ_s) is the s -th row of Ψ (the n_u -ordered identity matrix). Then, Eq. (7) is satisfied.

Proof. For the input constraint, refer to “Lemma 4” in [16].

Apply $\mathbf{z}(i+1|k) = \bar{\mathbf{C}}(k+i+1)[\mathbf{I} \quad 0][\Phi(i, k)\tilde{\mathbf{x}}(i|k) + \Gamma(i, k)\mathbf{w}(k+i)] + \bar{\mathbf{E}}(k+i+1)\mathbf{w}(k+i+1)$. Then

$$\max_{i \geq 0} |\Psi_s \mathbf{z}(i+1|k)|^2 \leq \max_{i \geq 0} \left\{ \frac{1}{(1-\eta_{3s})(1-\eta_{4s})} \left\| \Psi_s \bar{\mathbf{C}}(k+i+1)[\mathbf{I} \quad 0]\Phi(i, k)\mathbf{Q}^{\frac{1}{2}} \right\|^2 + \frac{1}{(1-\eta_{3s})\eta_{4s}} |\Psi_s \bar{\mathbf{C}}(k+i+1)[\mathbf{I} \quad 0]\Gamma(i, k)\mathbf{w}(k+i)|^2 + \frac{1}{\eta_{3s}} |\Psi_s \bar{\mathbf{E}}(k+i+1)\mathbf{w}(k+i+1)|^2 \right\} \quad (21)$$

Hence, if

$$\begin{aligned} & \bar{\psi}_s^2 - \frac{1}{\eta_{3s}} (\Psi_s \bar{\mathbf{E}}(k+i+1)\mathbf{w}(k+i+1))^2 - \frac{1}{(1-\eta_{3s})\eta_{4s}} [\Psi_s \bar{\mathbf{C}}(k+i+1)[\mathbf{I} \quad 0]\Gamma(i, k)\mathbf{w}(k+i)]^2 - \frac{1}{(1-\eta_{3s})(1-\eta_{4s})} \\ & \times \Psi_s \bar{\mathbf{C}}(k+i+1)[\mathbf{I} \quad 0]\Phi(i, k)\mathbf{Q}\Phi(i, k)^T[\mathbf{I} \quad 0]^T \bar{\mathbf{C}}(k+i+1)^T \Psi_s^T \geq 0, \quad s = 1, \dots, q, \end{aligned} \quad (22)$$

then $|\Psi_s \mathbf{z}(i+1|k)| \leq \bar{\psi}$ for all $i \geq 0$. Applying the Schur complement and convexity of the polytopic description, it is shown that Eq. (22), for all $i \geq 0$, is equivalent to

$$\sum_{h=1}^L \lambda_h(k+i+1) \left(\sum_{l=1}^L \sum_{j=1}^L \lambda_l(k+i) \lambda_j(k+i) \bar{\mathbf{T}}_{s,hlj}^{\text{OC}} \right) \geq 0, \quad s = 1, \dots, q, \quad (23)$$

$$\text{where } \bar{T}_{s,hlj}^{\text{oc}} = \begin{bmatrix} \mathbf{M} & \star & \star & \star \\ 0 & \mathbf{I} & \star & \star \\ 0 & 0 & \mathbf{I} & \star \\ \frac{1}{\sqrt{(1-\eta_{3s})(1-\eta_{4s})}} \Psi_s \bar{\mathbf{C}}_h \Phi_{lj}^1 & \frac{1}{\sqrt{(1-\eta_{3s})\eta_{4s}}} \Psi_s \bar{\mathbf{C}}_h \Gamma_l^1 \bar{\mathbf{w}}_j & \frac{1}{\sqrt{\eta_{3s}}} \Psi_s \bar{\mathbf{E}}_w \bar{\mathbf{w}}_h & \bar{\psi}_s^2 \end{bmatrix}. \text{Eq. (23) holds if}$$

$$\sum_{l=1}^L \sum_{j=1}^L \lambda_l(k+i) \lambda_j(k+i) \bar{T}_{s,hlj}^{\text{oc}} \geq 0, \quad h=1, \dots, L, \quad s=1, \dots, q \quad (24)$$

Applying the Schur complement, it is shown that Eq. (24) is equivalent to Eq. (20).

In Lemma 4, η_{1s} ($s=1, \dots, n_u$), η_{3s}, η_{4s} ($s=1, 2, \dots, q$) are the relaxation scalars. Lemma 4 keeps unchanged if we replace $\mathbf{w}(k)$ by $\mathbf{w}(k)\sigma(k)$ where $\sigma(k) \in \text{Co}\{-1, 1\}$ is unknown and persistent.

Summarizing Lemmas 2, 3 and 4, we approximate Eqs. (6)–(10) by

$$\min_{\boldsymbol{\theta}} \gamma, \text{ s.t. Eqs. (11)–(13), (16), (17), and (19), (20) and } \beta = 1, \mathbf{M} = \mathbf{Q}^{-1}, \alpha_{lj} = \alpha_{jl} \quad (25)$$

where $\boldsymbol{\theta} = \{\alpha_{lj} \ (l=1, \dots, L, j=1, \dots, L), \ \eta_{1s} \ (s=1, \dots, n_u), \ \eta_{3s}, \eta_{4s} \ (s=1, \dots, q), \ \gamma, \rho, \mathbf{A}_c, \mathbf{L}_c, \mathbf{F}_x, \mathbf{F}_y, \mathbf{M}_1, \mathbf{M}_3, \mathbf{Q}_1, \mathbf{Q}_3\}$. We set $\beta=1$ in Eq. (25). With the solution to Eq. (25), one can find

$$\beta_{\min} = \arg \min_{\beta} \beta, \text{ s.t. Eq. (16)}$$

If $\beta_{\min}(k) < 1$, then $\|\tilde{\mathbf{x}}(k+1)\|_{\mathbf{Q}(k)^{-1}} < \|\tilde{\mathbf{x}}(k)\|_{\mathbf{Q}(k)^{-1}}$; otherwise, it merely guarantees $\tilde{\mathbf{x}}(k+1) \in \varepsilon_{\mathbf{Q}(k)^{-1}}$. The problem Eq. (25) is non-convex, but its near-optimal solution can be found; refer to [4] and [14]. Note that [4] and [14] don't have η_{rs} . Since α_{lj} s and η_{rs} s can be line searched over $(0, 1)$, the existence of these scalars does not deny the near-optimality.

Theorem 1. For the system Eq. (1), the dynamic OFRMPC is adopted with $\{\mathbf{A}_c, \mathbf{L}_c, \mathbf{F}_x, \mathbf{F}_y\}(k)$ obtained via solving Eq. (25). If Eq. (25) is feasible

at time $k=0$, and Assumption 2 holds, and $\mathbf{M}_e(k)$ is refreshed according to

$$\mathbf{M}_e(k) = \mathbf{M}_1^*(k-1) \left[1 - \mathbf{x}_c(k)^T \left[\mathbf{M}_3^*(k-1) - \mathbf{E}_0 \mathbf{M}_1^*(k-1) \mathbf{E}_0^T \right] \mathbf{x}_c(k) \right]^{-1}, \quad (26)$$

at each time $k > 0$, then $\{\mathbf{z}', \mathbf{u}\}$ will converge to a neighborhood of $\{0, 0\}$, and constraints (4), (5) are satisfied for all $k \geq 0$.

Proof. Refer to “Theorem 1” in [16].

Choosing $\mathbf{z}' = \tilde{\mathbf{x}}$, then Theorem 1 claims stability of the augmented closed-loop system.

Remark 2. By choosing $\eta_{1s} = \eta_{4s} = \frac{1}{2}$ and $\eta_{3s} = \frac{1}{3}$, a simplified version for handling the physical constraints is recovered (see [2–4, 6]). The techniques in [2] and [6] utilize the polyhedral EEBs, so they do not guarantee the recursive feasibility of the main optimization problem.

3.2. A simplified synthesis approach of dynamic OFRMPC

By the simplified approach, we refer to the case when $\mathbf{L}_c$ and $\mathbf{F}_y$ are pre-specified. The pre-specification of $\{\mathbf{L}_c, \mathbf{F}_y\}$ is shown in Algorithm 1 in the sequel. Take the transformations $\hat{\mathbf{A}}_c = \mathbf{A}_c \mathbf{Q}_3$ and $\hat{\mathbf{F}}_x = \mathbf{F}_x \mathbf{Q}_3$. Take $\bar{\mathbf{C}}(k) = \mathbf{I}$. Applying a congruence transformation, via $\text{diag}\{\mathbf{Q}, \mathbf{I}\}$, it is shown that Eqs. (16), (17) hold if

$$\sum_{l=1}^L \lambda_l(k+i) \sum_{j=1}^L \lambda_j(k+i) \begin{bmatrix} (1-\alpha_{lj})\mathbf{Q} & \star & \star \\ 0 & \beta\alpha_{lj}\mathbf{I} & \star \\ \Sigma_{lj} & \Gamma_l^1 \bar{\mathbf{w}}_j & \mathbf{Q} \end{bmatrix} \geq 0, \quad (27)$$

$$\sum_{l=1}^L \lambda_l(k+i) \sum_{j=1}^L \lambda_j(k+i) \begin{bmatrix} \mathbf{Q} & \star & \star & \star \\ \Sigma_{lj} & \mathbf{Q} & \star & \star \\ \mathbf{W}_z^{1/2} \mathbf{Q} & 0 & \gamma \mathbf{I} & \star \\ \mathbf{W}_u^{1/2} \Sigma_j & 0 & 0 & \gamma \mathbf{I} \end{bmatrix} \geq 0, \quad (28)$$

where $\Sigma_{lj} = \begin{bmatrix} (\mathbf{A}_l + \mathbf{B}_l \mathbf{F}_y \mathbf{C}_j) \mathbf{Q}_1 + \mathbf{B}_l \hat{\mathbf{F}}_x \mathbf{E}_0 & (\mathbf{A}_l + \mathbf{B}_l \mathbf{F}_y \mathbf{C}_j) \mathbf{E}_0^T \mathbf{Q}_3 + \mathbf{B}_l \hat{\mathbf{F}}_x \\ \mathbf{L}_c \mathbf{C}_j \mathbf{Q}_1 + \hat{\mathbf{A}}_c \mathbf{E}_0 & \mathbf{L}_c \mathbf{C}_j \mathbf{E}_0^T \mathbf{Q}_3 + \hat{\mathbf{A}}_c \end{bmatrix}$, $\Sigma_j = \begin{bmatrix} \mathbf{F}_y \mathbf{C}_j \mathbf{Q}_1 + \hat{\mathbf{F}}_x \mathbf{E}_0 & \mathbf{F}_y \mathbf{C}_j \mathbf{E}_0^T \mathbf{Q}_3 + \hat{\mathbf{F}}_x \end{bmatrix}$. Applying the similar transformations on Eqs. (19), (20), we obtain the following conclusion.

Lemma 5. Suppose at time k , there exist scalars $\{\alpha_{lj}, \beta \leq 1, \eta_{rs}, \rho\}$ ($\alpha_{lj} = \alpha_{jl}$), matrices $\{\hat{\mathbf{A}}_c, \hat{\mathbf{F}}_x\}$ and symmetric matrices $\{\mathbf{Q}_1, \mathbf{Q}_3\}$ such that Eqs. (14), (15), (27) and (with $\mathbf{Q}_2 = \mathbf{Q}_3 \mathbf{E}_0$)

$$\begin{bmatrix} \mathbf{Q} & \star \\ \frac{1}{\sqrt{1-\eta_{1s}}} \xi_s \Sigma_j & \bar{u}_s^2 - \frac{1}{\eta_{1s}} (\xi_s \mathbf{F}_y \mathbf{E}_w \bar{\mathbf{w}}_j)^2 \end{bmatrix} \geq 0, \quad j=1, \dots, L, s=1, \dots, n_u \quad (29)$$

$$\sum_{l=1}^L \sum_{j=1}^L \lambda_l(k+i) \lambda_j(k+i) \left[\frac{\mathbf{Q}}{0} \quad \mathbf{I}^* \quad \mathbf{I}^* \right] \begin{bmatrix} 1 \\ \sqrt{(1-\eta_{3s})(1-\eta_{4s})} \end{bmatrix} \Psi_s \vec{\mathbf{C}}_h \Sigma_{lj}^1 \quad \frac{1}{\sqrt{(1-\eta_{3s})\eta_{4s}}} \Psi_s \vec{\mathbf{C}}_h \Gamma_l^1 \bar{\mathbf{w}}_j \quad \bar{\psi}_s^2 - \frac{1}{\eta_{3s}} \left(\Psi_s \vec{\mathbf{E}}_w \bar{\mathbf{w}}_h \right)^2 \geq 0, \quad h=1, \dots, L, \quad s=1, \dots, q \quad (30)$$

where $\Sigma_{lj}^1 = [(\mathbf{A}_l + \mathbf{B}_l \mathbf{F}_y \mathbf{C}_j) \mathbf{Q}_1 + \mathbf{B}_l \hat{\mathbf{F}}_x \mathbf{E}_0, (\mathbf{A}_l + \mathbf{B}_l \mathbf{F}_y \mathbf{C}_j) \mathbf{E}_0^T \mathbf{Q}_3 + \mathbf{B}_l \hat{\mathbf{F}}_x]$, and $\Psi_s (\xi_s)$ is the s -th row of Ψ (the n_u -ordered identity matrix). Then, Eq. (7) is satisfied.

In summary, we approximate Eqs. (6)–(10) by

$$\min_{\theta} \gamma, \quad \text{s.t.} \quad \text{Eqs. (14), (15), (27)–(30) and } \beta = 1, \mathbf{Q}_2 = \mathbf{Q}_3 \mathbf{E}_0, \alpha_{ij} = \alpha_{jl}, \quad (31)$$

where $\theta = \{\alpha_{ij} \ (l=1, \dots, L, \ j=1, \dots, L), \eta_{1s} \ (s=1, \dots, n_u), \eta_{3s}, \eta_{4s} \ (s=1, \dots, q), \gamma, \rho, \hat{\mathbf{A}}_c, \hat{\mathbf{F}}_x, \mathbf{Q}_1, \mathbf{Q}_3\}$. A comparison of the problems (25) and (31) are shown in Table 2. For Eq. (25), there are two sources of heavy computational burden (see [4,14]): (1) cone complementarity approach for achieving $\mathbf{M} = \mathbf{Q}^{-1}$, which often increases the computation burden for dozens of times; (2) iterative approach for minimizing γ , which often increases the computation burden for a few to hundreds of times. Therefore, Eq. (25) involves dozens to thousands of times larger computational burden than Eq. (31). For practical applications, it is more suitable to apply Eq. (31) than to apply Eq. (25).

Table 2
A comparison of Eqs. (25) and (31)

	Solution procedure	LMI decision variables	Initialization	Invariance	Optimality	Physical constraints	Line searches over (0,1)
(25)	Line searches, LMI toolbox, cone complementarity and iterative approaches	$\gamma, \rho, \mathbf{A}_c, \mathbf{L}_c, \mathbf{F}_x, \mathbf{F}_y, \mathbf{M}_1, \mathbf{M}_3, \mathbf{Q}_1, \mathbf{Q}_3 \ (\mathbf{M} = \mathbf{Q}^{-1})$	Eqs. (12), (13)	Eq. (16)	Eq. (17)	Eqs. (19), (20)	$\alpha_{ij} = \alpha_{ji}, \eta_{rs}$
(31)	Line searches, LMI toolbox	$\gamma, \rho, \hat{\mathbf{A}}_c, \hat{\mathbf{F}}_x, \mathbf{Q}_1, \mathbf{Q}_3$	Eqs. (14), (15)	Eq. (27)	Eq. (28)	Eqs. (29), (30)	$\alpha_{ij} = \alpha_{ji}, \eta_{rs}$

The closed-loop stability by applying Eq. (31) is similar to that by Eq. (25).

Corollary 1. For the system (1), the dynamic OFRMPC is adopted with $\{\mathbf{A}_c, \mathbf{F}_x\}(k)$ obtained via solving Eq. (31). If Eq. (31) is feasible at time $k=0$, and Assumption 2 holds, and $\mathbf{M}_e(k)$ is refreshed according to

$$\mathbf{M}_e(k) = \left[\mathbf{Q}_1^*(k-1) - \mathbf{E}_0^T \mathbf{Q}_3^*(k-1) \mathbf{E}_0 \right]^{-1} \left[1 - \mathbf{x}_c(k)^T \mathbf{Q}_3^*(k-1)^{-1} \mathbf{x}_c(k) \right]^{-1}, \quad (32)$$

at each time $k>0$, then $\{\bar{\mathbf{x}}, \mathbf{u}\}$ will converge to a neighborhood of $\{0,0\}$, and Eqs. (4), (5) are satisfied for all $k \geq 0$.

3.3. Pre-specifying relaxation scalars by norm-bounding technique

In both Eqs. (25) and (31), line searching η_{rs} considerably increases the computational burden. In [8], the similar scalars are eliminated (from on-line optimization) by applying the norm-bounding technique. In the following, we pre-specify η_{rs} by the extended norm-bounding technique.

Lemma 6. In the problem (31), the conditions (29), (30) hold if

$$\begin{bmatrix} \mathbf{Q} & \mathbf{I}^* \\ \xi_s \Sigma_j & \bar{u}_s^2 \end{bmatrix} \geq 0, \quad j=1, \dots, L, \quad s=1, \dots, n_u \quad (33)$$

$$\sum_{l=1}^L \lambda_l(k+i) \sum_{j=1}^L \lambda_j(k+i) \left[\frac{\mathbf{Q}}{\Psi_s \vec{\mathbf{C}}_h} \quad \mathbf{I}^* \quad \Sigma_{lj}^1 \bar{\psi}_s^2 \right] \geq 0, \quad h=1, \dots, L, \quad s=1, \dots, q, \quad (34)$$

where

$$\bar{u}_s = \bar{u}_s - \zeta_s^u, \quad (35)$$

$$\bar{\psi}_s = \bar{\psi}_s - \zeta_s^z - \bar{\zeta}_s^z, \quad (36)$$

$$\zeta_s^u = \max\{|\xi_s \mathbf{F}_y \mathbf{E}_w \bar{\mathbf{w}}_j| | j=1, \dots, L\}, \quad (37)$$

$$\zeta_s^z = \max\{|\Psi_s \vec{\mathbf{E}}_w \bar{\mathbf{w}}_h| | h=1, \dots, L\}, \quad (38)$$

$$\begin{aligned} \bar{\zeta}_s^z &= \arg \min_{\bar{\zeta}_s^z} \bar{\zeta}_s^z, \\ \text{s.t.} \quad & \sum_{l=1}^L \lambda_l(k+i) \sum_{j=1}^L \lambda_j(k+i) \left[\frac{\bar{\zeta}_s^z}{\Psi_s \vec{\mathbf{C}}_h \Gamma_l^1 \bar{\mathbf{w}}_j} \quad \mathbf{I}^* \quad \bar{\zeta}_s^z \right] \geq 0, \quad h=1, \dots, L. \end{aligned} \quad (39)$$

Proof. By taking ζ_s^u as in Eq. (37), and congruence transformation on Eq. (29) via $\text{diag}\{\mathbf{I}, \sqrt{1-\eta_{1s}}\}$, and $\bar{u}_s^2 = (1-\eta_{1s})[\bar{u}_s^2 - \frac{1}{\eta_{1s}}(\zeta_s^u)^2]$, it is

shown that Eq. (29) is guaranteed by Eq. (33). Then, the maximum $\tilde{u}_s$ is calculated by Eq. (35), taking $\eta_{1s} = \frac{\zeta_s^u}{\tilde{u}_s}$.

By taking $\{\zeta_s^z, \bar{\zeta}_s^z\}$ as in Eqs. (38), (39), and congruence transformation on Eq. (30) via $\text{diag}\{\mathbf{I}, \sqrt{(1-\eta_{3s})(1-\eta_{4s})}\}$, and applying the Schur complement, and taking $\tilde{\psi}_s^2 = (1-\eta_{4s})[(1-\eta_{3s})\tilde{\psi}_s^2 - \frac{1-\eta_{3s}}{\eta_{3s}}(\zeta_s^z)^2 - \frac{1}{\eta_{4s}}(\bar{\zeta}_s^z)^2]$, it is shown that Eq. (30) is guaranteed by Eq. (34). Then, the maximum $\tilde{\psi}_s$ is calculated by Eq. (36), taking $\eta_{3s} = \frac{\zeta_s^z}{\tilde{\psi}_s}$ and $\eta_{4s} = \frac{\bar{\zeta}_s^z}{\tilde{\psi}_s - \zeta_s^z}$.

Algorithm 1. (Fixing $\{\mathbf{F}_y, \mathbf{L}_c\}$ in Eq. (31), and $\{\alpha_{ij} = \alpha, \eta_{rs}\}$ in Eqs. (25) and (31))

(1) Solve (cone complementarity and iterative approaches, as in [4] and [14])

$$\min_{\alpha, \gamma, \mathbf{A}_c, \mathbf{L}_c, \mathbf{F}_x, \mathbf{F}_y, \mathbf{M}_1, \mathbf{M}_3, \mathbf{Q}_1, \mathbf{Q}_3} \gamma, \quad \text{s.t.} \quad (16), (17) \quad \text{and} \quad \beta = 1, \mathbf{M} = \mathbf{Q}^{-1}, \alpha_{ij} = \alpha, \quad (40)$$

to obtain $\{\mathbf{F}_y, \mathbf{L}_c, \alpha\}$. Thus, $\alpha_{ij} = \alpha$ in Eq. (25), and $\{\alpha_{ij} = \alpha, \mathbf{F}_y, \mathbf{L}_c\}$ in Eq. (31), are fixed.

(2) Apply Eqs. (37), (39) to obtain $\{\zeta_s^u, \zeta_s^z, \bar{\zeta}_s^z\}$, and taking $\eta_{1s} = \frac{\zeta_s^u}{\tilde{u}_s}$, $\eta_{3s} = \frac{\zeta_s^z}{\tilde{\psi}_s}$ and $\eta_{4s} = \frac{\bar{\zeta}_s^z}{\tilde{\psi}_s - \zeta_s^z}$. Thus, η_{rs} s in Eqs. (25) and (31) are fixed.

The problem (40) omits the physical constraints. Hence, in Algorithm 1, the local optimality of $\{\alpha_{ij} = \alpha, \eta_{rs}, \mathbf{F}_y, \mathbf{L}_c\}$ is preserved.

4. Applying to Different System Assumptions

Firstly, Eqs. (25) and (31) are applied to a set of special system assumptions. Then, taking [8] as an example, it is shown that some previous approaches are special cases.

4.1. Applying to a set of special system assumptions

Assume that $\mathbf{D}(k)\mathbf{w}(k) = \bar{\mathbf{D}}(k)\mathbf{v}'(k)$, $\mathbf{E}(k)\mathbf{w}(k) = \bar{\mathbf{E}}(k)\mathbf{w}'(k)$ and $\mathbf{w} = [\mathbf{v}'^T \quad \mathbf{w}'^T]^T$. Rewrite $\{\bar{\mathbf{D}}, \bar{\mathbf{E}}, \mathbf{v}', \mathbf{w}', \mathbf{w}\}$ as $\{\mathbf{D}, \mathbf{E}, \mathbf{v}, \mathbf{w}, \bar{\mathbf{v}}\}$. For this equivalent case, the augmented prediction model Eq. (3) becomes.

$$\bar{\mathbf{x}}(i+1|k) = \Phi(i, k)\bar{\mathbf{x}}(i|k) + \tilde{\Gamma}(i, k)\bar{\mathbf{v}}(k+i), \quad i \geq 0, \quad \bar{\mathbf{x}}(0|k) = \bar{\mathbf{x}}(k), \quad (41)$$

where $\tilde{\Gamma}(i, k) = \begin{bmatrix} \mathbf{D}(k+i) & \mathbf{B}(k+i)\mathbf{F}_y(k)\mathbf{E}(k+i) \\ 0 & \mathbf{L}_c(k)\mathbf{E}(k+i) \end{bmatrix}$. In [8], the following assumptions are given for Eq. (41).

Assumption 3. $[\mathbf{A}|\mathbf{B}|\mathbf{C}](k) \in \Omega^1 := \text{Co}\{[\mathbf{A}_l|\mathbf{B}_l|\mathbf{C}_l] \mid l=1, \dots, L\}$, i.e., there exist nonnegative coefficients $\lambda_l(k)$ such that $\sum_{l=1}^L \lambda_l(k) = 1$ and $[\mathbf{A}|\mathbf{B}|\mathbf{C}](k) = \sum_{l=1}^L \lambda_l(k)[\mathbf{A}_l|\mathbf{B}_l|\mathbf{C}_l]$. Moreover, $[\mathbf{D}|\mathbf{E}](k) \in \Omega^2 := \text{Co}\{[\mathbf{D}_l|\mathbf{E}_l] \mid l=1, \dots, p\}$.

Assumption 4. $\mathbf{v}(k) \in \mathbb{V} := \text{Co}\{\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \dots, \bar{\mathbf{v}}_{m_v}\} \supseteq 0$ and $\mathbf{w}(k) \in \mathbb{W} := \text{Co}\{\bar{\mathbf{w}}_1, \bar{\mathbf{w}}_2, \dots, \bar{\mathbf{w}}_{m_w}\} \supseteq 0$, for all $k \geq 0$.

By taking $\Omega = \Omega^1 \times \Omega^2 \times \mathbb{V} \times \mathbb{W}$, Assumptions 3, 4 are recovered from Assumption 1. With Assumptions 3, 4, we can give an approach of OFRMPC parallel to Eq. (25) (where $\mathbf{z} = \bar{\mathbf{x}}$), i.e.,

$$\min_{\Theta} \gamma, \quad \text{s.t.} \quad \text{Eq. (11)–(13), (17), (43)–(45)} \quad \text{and} \quad \beta = 1, \mathbf{M} = \mathbf{Q}^{-1}, \alpha_{ij} = \alpha_{jl} \quad (42)$$

$$\sum_{l=1}^L \lambda_l(k+i) \sum_{j=1}^L \lambda_j(k+i) \begin{bmatrix} (1-\alpha_{ij})\mathbf{M} & \star & \star \\ 0 & \beta\alpha_{ij} & \star \\ \Phi_{ij} & \begin{bmatrix} \mathbf{D}_l\bar{\mathbf{v}}_r + \mathbf{B}_l\mathbf{F}_y\mathbf{E}_l\bar{\mathbf{w}}_\tau \\ \mathbf{L}_c\mathbf{E}_l\bar{\mathbf{w}}_\tau \end{bmatrix} & \mathbf{Q} \end{bmatrix} \geq 0, \quad t=1, \dots, p, \quad r=1, \dots, m_v, \quad \tau=1, \dots, m_w, \quad (43)$$

$$\begin{bmatrix} \mathbf{M} & \star & \star \\ 0 & 1 & \star \\ \frac{1}{\sqrt{1-\eta_{1s}}}\xi_s[\mathbf{F}_y\mathbf{C}_j \quad \mathbf{F}_x] & \frac{1}{\sqrt{\eta_{1s}}}\xi_s\mathbf{F}_y\mathbf{E}_t\bar{\mathbf{w}}_\tau & \bar{u}_s^2 \end{bmatrix} \geq 0, \quad j=1, \dots, L, \quad t=1, \dots, p, \quad \tau=1, \dots, m_w, \quad s=1, \dots, n_u, \quad (44)$$

$$\sum_{l=1}^L \sum_{j=1}^L \lambda_l(k+i) \lambda_j(k+i) \begin{bmatrix} \mathbf{M} & \star & \star \\ 0 & 1 & \star \\ \frac{1}{\sqrt{1-\eta_{2s}}}\Psi_s\Phi_{ij}^1 & \frac{1}{\sqrt{\eta_{2s}}}\Psi_s(\mathbf{D}_l\bar{\mathbf{v}}_r + \mathbf{B}_l\mathbf{F}_y\mathbf{E}_l\bar{\mathbf{w}}_\tau) & \bar{\psi}_s^2 \end{bmatrix} \geq 0, \quad t=1, \dots, p, \quad r=1, \dots, m_v, \quad \tau=1, \dots, m_w, \quad s=1, \dots, q \quad (45)$$

where $\Theta = \{\alpha_{ij} \mid (l=1, \dots, L, j=1, \dots, L), \eta_{1s} \mid (s=1, \dots, n_u), \eta_{2s} \mid (s=1, \dots, q), \gamma, \rho, \mathbf{A}_c, \mathbf{L}_c, \mathbf{F}_x, \mathbf{F}_y, \mathbf{M}_1, \mathbf{M}_3, \mathbf{Q}_1, \mathbf{Q}_3\}$. The closed-loop stability holds the same as Eq. (25).

With Assumptions 3, 4, we can also give a simplified approach of OFRMPC parallel to Eq. (31) (where $\mathbf{z}=\mathbf{x}$), i.e.,

$$\min_{\theta} \gamma, \text{ s.t. Eqs. (14), (15), (28), (47)–(49) and } \beta = 1, \mathbf{Q}_2 = \mathbf{Q}_3 \mathbf{E}_0, \alpha_{lj} = \alpha_{jl}, \quad (46)$$

$$\sum_{l=1}^L \lambda_l(k+i) \sum_{j=1}^L \lambda_j(k+i) \begin{bmatrix} (1-\alpha_{lj})\mathbf{Q} & \star & \star \\ 0 & \beta\alpha_{lj} & \star \\ \Sigma_{lj} & \begin{bmatrix} \mathbf{D}_t \bar{\mathbf{v}}_r + \mathbf{B}_l \mathbf{F}_y \mathbf{E}_t \bar{\mathbf{w}}_\tau \\ \mathbf{L}_c \mathbf{E}_t \bar{\mathbf{w}}_\tau \end{bmatrix} & \mathbf{Q} \end{bmatrix} \geq 0, \quad (47)$$

$t = 1, \dots, p, \quad r = 1, \dots, m_v, \quad \tau = 1, \dots, m_w$

$$\left[\frac{1}{\sqrt{1-\eta_{1s}}} \mathbf{Q} \quad \star \right. \\ \left. \xi_s \Sigma_j \quad \bar{u}_s^2 - \frac{1}{\eta_{1s}} (\xi_s \mathbf{F}_y \mathbf{E}_t \bar{\mathbf{w}}_\tau)^2 \right] \geq 0, \quad (48)$$

$j = 1, \dots, L, \quad t = 1, \dots, p, \quad \tau = 1, \dots, m_w, \quad s = 1, \dots, n_u,$

$$\sum_{l=1}^L \sum_{j=1}^L \lambda_l(k+i) \lambda_j(k+i) \begin{bmatrix} \mathbf{Q} & \star & \star \\ 0 & 1 & \star \\ \frac{1}{\sqrt{1-\eta_{2s}}} \Psi_s \Sigma_{lj}^1 & \frac{1}{\sqrt{\eta_{2s}}} \Psi_s (\mathbf{D}_t \bar{\mathbf{v}}_r + \mathbf{B}_l \mathbf{F}_y \mathbf{E}_t \bar{\mathbf{w}}_\tau) & \bar{\psi}_s^2 \end{bmatrix} \geq 0, \quad (49)$$

$t = 1, \dots, p, \quad r = 1, \dots, m_v, \quad \tau = 1, \dots, m_w, \quad s = 1, \dots, q$

where $\theta = \{\alpha_{lj}(l=1, \dots, L, j=1, \dots, L), \eta_{1s}(s=1, \dots, n_u), \eta_{2s}(s=1, \dots, q), \gamma, \rho, \hat{\mathbf{A}}_c, \hat{\mathbf{F}}_x, \mathbf{Q}_1, \mathbf{Q}_3\}$. The conclusion in Corollary 1 holds when Eq. (31) is replaced by Eq. (46).

Lemma 7. In the problem (46), the conditions (48), (49) are guaranteed by Eqs. (33), (34), where $\{\bar{u}_s, \bar{\psi}_s\}$ are calculated by

$$\bar{u}_s = \bar{u}_s - \zeta_s^u, \quad \bar{\psi}_s = \bar{\psi}_s - \zeta_s^x, \quad (50)$$

with

$$\zeta_s^u = \max\{|\xi_s \mathbf{F}_y \mathbf{E}_t \bar{\mathbf{w}}_\tau| | t=1, \dots, p, \quad \tau=1, \dots, m_w\}, \quad (51)$$

$$\zeta_s^x = \max\{|\Psi_s (\mathbf{D}_t \bar{\mathbf{v}}_r + \mathbf{B}_l \mathbf{F}_y \mathbf{E}_t \bar{\mathbf{w}}_\tau)| | l=1, \dots, L, \quad t=1, \dots, p, \quad r=1, \dots, m_v, \quad \tau=1, \dots, m_w\}. \quad (52)$$

4.2. Reformulation of a previous approach as a special case

Using the notations in this paper, let us rewrite the optimization problem of [8] ($n_x = n_{x_c}$). The problem (8)–(13) in [8] is rewritten as.

$$\min_{\gamma, \mathbf{Q}, \mathbf{A}_c, \mathbf{F}_x} \max_{\mathbf{A}|\mathbf{B}|\mathbf{C}|\mathbf{D}|\mathbf{E}|\mathbf{F}|\mathbf{G}|\mathbf{H}|\mathbf{I}|\mathbf{J}|\mathbf{K}|\mathbf{L}|\mathbf{M}|\mathbf{N}|\mathbf{O}|\mathbf{P}|\mathbf{Q}|\mathbf{R}|\mathbf{S}|\mathbf{T}|\mathbf{U}|\mathbf{V}|\mathbf{W}|\mathbf{X}|\mathbf{Y}|\mathbf{Z}|\mathbf{A}|\mathbf{B}|\mathbf{C}|\mathbf{D}|\mathbf{E}|\mathbf{F}|\mathbf{G}|\mathbf{H}|\mathbf{I}|\mathbf{J}|\mathbf{K}|\mathbf{L}|\mathbf{M}|\mathbf{N}|\mathbf{O}|\mathbf{P}|\mathbf{Q}|\mathbf{R}|\mathbf{S}|\mathbf{T}|\mathbf{U}|\mathbf{V}|\mathbf{W}|\mathbf{X}|\mathbf{Y}|\mathbf{Z}} \gamma, \quad (53)$$

$$\text{s.t. (18), } -\bar{\mathbf{u}} \leq \mathbf{u}(i|k) \leq \bar{\mathbf{u}}, -\bar{\mathbf{e}} \leq \mathbf{e}(i|k) \leq \bar{\mathbf{e}}, -\bar{\psi} \leq \Psi \mathbf{x}(i+1|k) \leq \bar{\psi} \quad (54)$$

$$\|\bar{\mathbf{x}}_u(i|k)\|_{\mathbf{Q}^{-1}}^2 - \|\bar{\mathbf{x}}_u(i+1|k)\|_{\mathbf{Q}^{-1}}^2 \geq 1/\gamma \left[\|\mathbf{x}_{c,u}(i|k)\|_{\mathbf{W}_x}^2 + \|\mathbf{e}_u(i|k)\|_{\mathbf{W}_e}^2 + \|\mathbf{u}_u(i|k)\|_{\mathbf{W}_u}^2 \right], \quad (55)$$

where $i \geq 0$, $\mathbf{e}_u = \mathbf{x}_u - \mathbf{x}_{c,u}$, $\mathbf{W}_x > 0$, $\mathbf{W}_e > 0$ and $\mathbf{W}_u > 0$. $\bar{\mathbf{e}} = [\bar{e}_1, \bar{e}_2, \dots, \bar{e}_{n_x}]^T$ is pre-specified such that $\mathbf{e}(0) \in \{\mathbf{e} | -\bar{\mathbf{e}} \leq \mathbf{e} \leq \bar{\mathbf{e}}\} = \text{Co}\{\mathbf{e}_1^v, \mathbf{e}_2^v, \dots, \mathbf{e}_{2^{n_x}}^v\}$.

The work [8] applies the dilation technique. Take $\tilde{\mathbf{T}} = \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 \\ \mathbf{T}_3 & \mathbf{T}_3 \end{bmatrix}$, $\mathbf{M}_c = \mathbf{A}_c \mathbf{T}_3$ and $\mathbf{Y} = \mathbf{F}_x \mathbf{T}_3$, where $\tilde{\mathbf{T}}$ is the dilation matrix. Define

$$\mathbf{S} = \begin{bmatrix} \mathbf{T}_1^T + \mathbf{T}_1 - \mathbf{Q}_1 & \star \\ \mathbf{T}_2^T + \mathbf{T}_3 - \mathbf{Q}_2 & \mathbf{T}_3^T + \mathbf{T}_3 - \mathbf{Q}_3 \end{bmatrix}. \text{ Then, we approximate the problems (53)–(55) as}$$

$$\min_{\theta} \gamma, \text{ s.t. Eqs. (57)–(62) and } \beta = 1, \alpha_{lj} = \alpha_{jl}, \quad (56)$$

$$\begin{bmatrix} 1 & \star & \star \\ \mathbf{e}_j^v & \mathbf{Q}_1 - \mathbf{Q}_2 - \mathbf{Q}_2^T + \mathbf{Q}_3 & \star \\ \mathbf{x}_c(k) & \mathbf{Q}_2 - \mathbf{Q}_3 & \mathbf{Q}_3 \end{bmatrix} \geq 0, \quad j = 1, \dots, 2^{n_x}, \quad (57)$$

$$\sum_{l=1}^L \lambda_l(k+i) \sum_{j=1}^L \lambda_j(k+i) \begin{bmatrix} (1-\alpha_{lj})\mathbf{S} & \star & \star \\ 0 & \beta\alpha_{lj} & \star \\ \Sigma_{lj} & \mathbf{N}_{thr\tau} & \mathbf{Q} \end{bmatrix} \geq 0, \quad (58)$$

$$\sum_{l=1}^L \lambda_l(k+i) \sum_{j=1}^L \lambda_j(k+i) \begin{bmatrix} \mathbf{S} & \star & \star \\ \Sigma_{lj} & \star & \star \\ \begin{bmatrix} \mathbf{W}_e^{1/2} [\mathbf{T}_1 - \mathbf{T}_3, \mathbf{T}_2 - \mathbf{T}_3] \\ \mathbf{W}_x^{1/2} [\mathbf{T}_3, \mathbf{T}_3] - \mathbf{W}_e^{1/2} [\mathbf{T}_1 - \mathbf{T}_3, \mathbf{T}_2 - \mathbf{T}_3] \\ \mathbf{W}_u^{1/2} \Sigma_j \end{bmatrix} & 0 & \gamma \mathbf{I} \end{bmatrix} \geq 0, \quad (59)$$

$$\sum_{l=1}^L \lambda_l(k+i) \sum_{j=1}^L \lambda_j(k+i) \left[\begin{array}{ccc} \mathbf{S} & \star & \star \\ 0 & 1 & \star \\ \frac{1}{\sqrt{1-\eta_{ls}}} \xi_s^c \boldsymbol{\theta}_{lj} & \frac{1}{\sqrt{\eta_{ls}}} \xi_s^c [\mathbf{D}_l \bar{\mathbf{v}}_r + (\mathbf{B}_l \mathbf{F}_y - \mathbf{L}_c) \mathbf{E}_t \bar{\mathbf{w}}_\tau] & \bar{\mathbf{e}}_s^2 \end{array} \right] \geq 0, \quad (60)$$

$$s = 1, \dots, n_x, \\ \boldsymbol{\theta}_{lj} = [(\mathbf{A}_l + \mathbf{B}_l \mathbf{F}_y \mathbf{C}_j - \mathbf{L}_c \mathbf{C}_j) \mathbf{T}_1 - \mathbf{M}_c + \mathbf{B}_l \mathbf{Y}, (\mathbf{A}_l + \mathbf{B}_l \mathbf{F}_y \mathbf{C}_j - \mathbf{L}_c \mathbf{C}_j) \mathbf{T}_2 - \mathbf{M}_c + \mathbf{B}_l \mathbf{Y}],$$

$$\left[\begin{array}{cc} \mathbf{S} & \star \\ \frac{1}{\sqrt{1-\eta_{ls}}} \xi_s \boldsymbol{\Sigma}_j & \bar{\mathbf{u}}_s^2 - \frac{1}{\eta_{ls}} (\xi_s \mathbf{F}_y \mathbf{E}_t \bar{\mathbf{w}}_\tau)^2 \end{array} \right] \geq 0, \quad j = 1, \dots, L, \quad s = 1, \dots, n_u, \quad (61)$$

$$\sum_{l=1}^L \sum_{j=1}^L \lambda_l(k+i) \lambda_j(k+i) \left[\begin{array}{ccc} \mathbf{S} & \star & \star \\ 0 & 1 & \star \\ \frac{1}{\sqrt{1-\eta_{2s}}} \boldsymbol{\Psi}_s \boldsymbol{\Sigma}_{lj}^1 & \frac{1}{\sqrt{\eta_{2s}}} \boldsymbol{\Psi}_s (\mathbf{D}_l \bar{\mathbf{v}}_r + \mathbf{B}_l \mathbf{F}_y \mathbf{E}_t \bar{\mathbf{w}}_\tau) & \bar{\psi}_s^2 \end{array} \right] \geq 0, \quad (62)$$

$$s = 1, \dots, q,$$

where $\boldsymbol{\theta} = \{\alpha_{lj}(l = 1, \dots, L, j = 1, \dots, L), \eta_{ls}(s = 1, \dots, n_u), \eta_{2s}(s = 1, \dots, q), \eta_{ls}(s = 1, \dots, n_x), \gamma, \mathbf{M}_c, \mathbf{Y}, \mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3\}$, $t = 1, \dots, p$, $r = 1, \dots, m_v$, $\tau = 1, \dots, m_w$, $\mathbf{N}_{lrr} = \begin{bmatrix} \mathbf{D}_l \bar{\mathbf{v}}_r + \mathbf{B}_l \mathbf{F}_y \mathbf{E}_t \bar{\mathbf{w}}_\tau \\ \mathbf{L}_c \mathbf{E}_t \bar{\mathbf{w}}_\tau \end{bmatrix}$, $\boldsymbol{\Sigma}_j = [\mathbf{F}_y \mathbf{C}_j \mathbf{T}_1 + \mathbf{Y} \quad \mathbf{F}_y \mathbf{C}_j \mathbf{T}_2 + \mathbf{Y}]$, and $\boldsymbol{\Sigma}_{lj} = \begin{bmatrix} (\mathbf{A}_l + \mathbf{B}_l \mathbf{F}_y \mathbf{C}_j) \mathbf{T}_1 + \mathbf{B}_l \mathbf{Y} & (\mathbf{A}_l + \mathbf{B}_l \mathbf{F}_y \mathbf{C}_j) \mathbf{T}_2 + \mathbf{B}_l \mathbf{Y} \\ \mathbf{L}_c \mathbf{C}_j \mathbf{T}_1 + \mathbf{M}_c & \mathbf{L}_c \mathbf{C}_j \mathbf{T}_2 + \mathbf{M}_c \end{bmatrix}$ (with abuse of notations), and ξ_s^c is the s -th row of the n_x -ordered identity matrix. By taking congruence transformations, via $\text{diag}\{\tilde{\mathbf{T}}, \mathbf{I}\}$, and applying the fact $\tilde{\mathbf{T}}^T + \tilde{\mathbf{T}} - \mathbf{Q} \leq \tilde{\mathbf{T}}^T \mathbf{Q}^{-1} \tilde{\mathbf{T}}$ (dilation technique), it is shown that Eqs. (43)–(45) become Eqs. (58) and (61), (62). For EEC and the optimality conditions, similar dilation trick is applied. With Eq. (18), EEC $-\bar{\mathbf{e}} \leq \mathbf{e}(i|k) \leq \bar{\mathbf{e}}$ is guaranteed by Eq. (60), and Eq. (18) by Eqs. (57), (58). Similar to Eq. (28), the optimality condition (55) is guaranteed by Eq. (59).

Remark 3. Introducing $\tilde{\mathbf{T}}$ changes the numerical solutions, and can be influential for the receding-horizon implementation. However, $\tilde{\mathbf{T}}$ has an adverse effect on feasibility. The congruence transformation does not affect the feasibility. If Eq. (56) is infeasible, but becomes feasible when $\tilde{\mathbf{T}}^T + \tilde{\mathbf{T}} - \mathbf{Q}$ is replaced by $\tilde{\mathbf{T}}^T \mathbf{Q}^{-1} \tilde{\mathbf{T}}$, then it is the special structure of $\tilde{\mathbf{T}}$ which leads to infeasibility; one cannot take $\tilde{\mathbf{T}} = \mathbf{Q}$ (see [18]) to deny this fact, since Eq. (56) does not take $\mathbf{Q}_2 = \mathbf{Q}_3$. Only when $\mathbf{Q}_2 = \mathbf{Q}_3$ is specified, the above special structure of $\tilde{\mathbf{T}}$ will not affect the feasibility.

Proposition 2. Problem (56) is less conservative than Eq. (22) in [8] (which approximates the problems (8)–(13) in [8]), i.e., the latter is a special case of the former.

Proof. Refer to Proposition 1 in [17].

We cannot show that Eq. (56) (or Eq. (22) in [8]) is a special case of Eq. (46). Anyway, Eq. (56) (and Eq. (22) in [8]) utilizes the full $\mathbf{Q}$ and $\tilde{\mathbf{T}}$. Besides the full $\mathbf{Q}$ and $\tilde{\mathbf{T}}$, Eq. (46) is better than Eq. (56). As compared with Eq. (46), there are two weaknesses in Eq. (56): (1) the loss of recursive feasibility; (2) the intrusion of conservatism (both in feasibility and optimality) by imposing EEC. In order to guarantee recursive feasibility, Eq. (56) can be modified with $\mathbf{Q}_2 = \mathbf{Q}_3 \mathbf{E}_0$ and Lemma 2; in such a case, by further taking $\tilde{\mathbf{T}} = \mathbf{Q}$ we can say that Eq. (46) includes the modified optimization problem as a special case.

Indeed, it is easier to show that the techniques in this paper include the counterparts in [4] and [6] as special cases.

Remark 4. The approaches in this paper are computationally formidable for moderate or large dimensional systems. One can use the off-line MPC, which off-line optimizes a series of $\{\mathbf{A}_c, \mathbf{L}_c, \mathbf{F}_x, \mathbf{F}_y, \mathbf{Q}\}$ corresponding to a series of off-line selected $\{\mathbf{M}_e, \mathbf{x}_c\}$ (stored in the look-up table), and on-line (at each time $k \geq 0$), chooses a set of $\{\mathbf{A}_c, \mathbf{L}_c, \mathbf{F}_x, \mathbf{F}_y\}(k)$ (among the look-up table) according to the real-time $\{\mathbf{x}_c, \mathbf{M}_e, \rho\}(k)$. This is a usual procedure of gain-scheduling control, which is not detailed here.

5. Illustrative Example

We obtain the model parameters by handling the model of a continuous stirred tank reactor (CSTR) for an exothermic, irreversible reaction. For the physical meaning of CSTR model, refer to [13]. The model parameters corresponding to Eq. (1) are: $L = 4$,

$$\mathbf{A}_1 = \begin{bmatrix} 0.8227 & -0.00168 \\ 6.1233 & 0.9367 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 0.9654 & -0.00182 \\ -0.6759 & 0.9433 \end{bmatrix}, \\ \mathbf{A}_3 = \begin{bmatrix} 0.8895 & -0.00294 \\ 2.9447 & 0.9968 \end{bmatrix}, \mathbf{A}_4 = \begin{bmatrix} 0.8930 & -0.00062 \\ 2.7738 & 0.8864 \end{bmatrix}, \\ \mathbf{B}_1 = \begin{bmatrix} -0.000092 \\ 0.1014 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} -0.000097 \\ 0.1016 \end{bmatrix}, \mathbf{B}_3 = \begin{bmatrix} -0.000157 \\ 0.1045 \end{bmatrix}, \\ \mathbf{B}_4 = \begin{bmatrix} -0.000034 \\ 0.0986 \end{bmatrix}, \mathbf{C}_l = [0 \quad 1], \mathbf{D}_l = \begin{bmatrix} 0.00223 & 0 \\ 0 & 0.0564 \end{bmatrix}, \\ \mathbf{E}_l = [0.5 \quad 0.5], l = 1, \dots, 4.$$

The physical constraints include $|u| \leq 10$, $|\mathbf{x}_1| \leq 0.5$ and $|y| \leq 10$. Moreover, $[\mathbf{A}|\mathbf{B}|\mathbf{C}|\mathbf{D}|\mathbf{E}](k) = \sum_{l=1}^4 \lambda_l(k) [\mathbf{A}_l|\mathbf{B}_l|\mathbf{C}_l|\mathbf{D}_l|\mathbf{E}_l]$ where

$$\lambda_1 = \frac{1}{2} \frac{\varphi_1(y) - \varphi_1(-\bar{\psi}^y)}{\varphi_1(\bar{\psi}^y) - \varphi_1(-\bar{\psi}^y)}, \lambda_2 = \frac{1}{2} \frac{\varphi_1(\bar{\psi}^y) - \varphi_1(y)}{\varphi_1(\bar{\psi}^y) - \varphi_1(-\bar{\psi}^y)}, \\ \lambda_3 = \frac{1}{2} \frac{\varphi_2(y) - \varphi_2(-\bar{\psi}^y)}{\varphi_2(\bar{\psi}^y) - \varphi_2(-\bar{\psi}^y)}, \lambda_4 = \frac{1}{2} \frac{\varphi_2(\bar{\psi}^y) - \varphi_2(y)}{\varphi_2(\bar{\psi}^y) - \varphi_2(-\bar{\psi}^y)},$$

$\varphi_1(y) = 7.2 \times 10^{10} e^{\frac{8750}{y+350}}$, $\varphi_2(y) = 3.6 \times 10^{10} (e^{\frac{8750}{y+350}} - e^{\frac{8750}{350}})/y$, with $\bar{\psi}^y = 10$. $\mathbf{w}(k) \in \text{Co}\{\bar{\mathbf{w}}_1, \bar{\mathbf{w}}_2, \bar{\mathbf{w}}_3, \bar{\mathbf{w}}_4\}$, where $\bar{\mathbf{w}}_1 = \frac{1}{\sqrt{2}} [-1 \quad -1]$, $\bar{\mathbf{w}}_2 = \frac{1}{\sqrt{2}} [-1 \quad 1]$, $\bar{\mathbf{w}}_3 = \frac{1}{\sqrt{2}} [1 \quad -1]$ and $\bar{\mathbf{w}}_4 = \frac{1}{\sqrt{2}} [1 \quad 1]$. Technically, in Eqs. (25) and (31), replace $\{\mathbf{D}_w \bar{\mathbf{w}}_j, \mathbf{E}_w \bar{\mathbf{w}}_j, \bar{\mathbf{E}}_w \bar{\mathbf{w}}_h\}$ by $\{\mathbf{D}_l \bar{\mathbf{w}}_\tau, \mathbf{E}_j \bar{\mathbf{w}}_\tau, \bar{\mathbf{E}}_h \bar{\mathbf{w}}_\tau\}$ for all $\tau = 1, \dots, 4$ and all $\tau' = 1, \dots, 4$.

At $k=0$, Eq. (56) is infeasible for any $\mathbf{x}_c(0)$ and $\bar{\epsilon}$, which means that the procedure in [8] cannot be applied. In the sequel we simulate the following three methods:

- Mthd 1: applying Eq. (25), where η_{rs} are pre-specified as $\eta_{1s} = \eta_{4s} = \frac{1}{2}$ and $\eta_{3s} = \frac{1}{3}$;
 Mthd 2: similar to Mthd1, but η_{rs} are optimized according to Algorithm 1;
 Mthd 3: applying Eq. (31), where η_{rs} are optimized according to Algorithm 1.

In Mthd1–Mthd3, for Eq. (25) and Algorithm 1, choose $\{N_0, d, d^0, \kappa\} = \{200, 0, 0, 0.98\}$ (see [4,14]: N_0 is the maximum allowable amount of iterations in the cone complementarity approach; d and d^0 are the complexity index and the maximum complexity index for handling the DbCCs, respectively; κ is the accuracy for minimizing γ). Mthd1 and Mthd2 differ in whether or not η_{rs} are optimized. Mthd2 and Mthd3 differ in whether or not $\{F_y, L_c\}$ are on-line optimized.

Choose $E_0 = I$ and $M_e(0) = \begin{bmatrix} 100 & 0 \\ 0 & 0.25 \end{bmatrix}$. In selecting $\mathbf{x}_c(0)$, the basic requirement $\mathbf{x}(0) - E_0^T \mathbf{x}_c(0) \in \varepsilon_{M_e(0)}$ should be satisfied. $\mathbf{w}(k)$ is randomly generated. Apply the same disturbance sequence for all the methods. The following two sets of simulations are reported.

- (1) Choose $W_z = 16$ and $W_u = 9$. By applying Algorithm 1, we obtain $\alpha = 0.01$, $L_c = [0.0002, 0.0096]^T$ and $F_y = -1.1372$. The regions of attraction by applying Mthd1–Mthd3 are depicted in Fig. 1. Take $\mathbf{x}_c(0) = \frac{1}{\sqrt{2}}[-0.22, 4.8]^T$ and $\mathbf{x}(0) = \frac{1}{\sqrt{2}}[-0.32, 6.8]^T$.

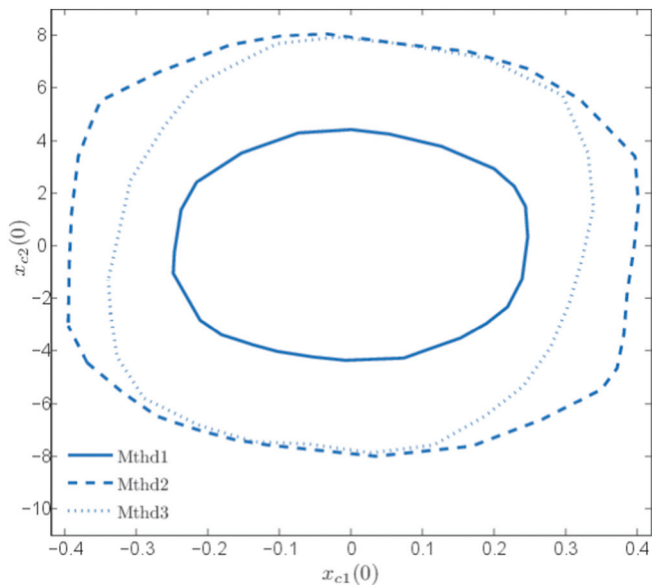


Fig. 1. The regions of attraction, set (1).

- (2) Choose $W_z = 25$ and $W_u = 1$. By applying Algorithm 1, we obtain $\alpha = 0.01$, $L_c = [0.0002, 0.0030]^T$ and $F_y = -1.9877$. The regions of attraction by applying Mthd1–Mthd3 are depicted in Fig. 2. Take $\mathbf{x}_c(0) = \frac{1}{\sqrt{2}}[0.05, 6]^T$ and $\mathbf{x}(0) = \frac{1}{\sqrt{2}}[0.15, 8]^T$.

Here, the region of attraction, say $\mathbb{X}_c$, is the region of $\mathbf{x}_c(0)$ such that, whenever $\mathbf{x}_c(0) \in \mathbb{X}_c$, the optimization problem is feasible. Some further results are shown in Figs. 3–6. The evolution of γ , as in Figs. 3, 4, represents the control performance since we consider the min–max

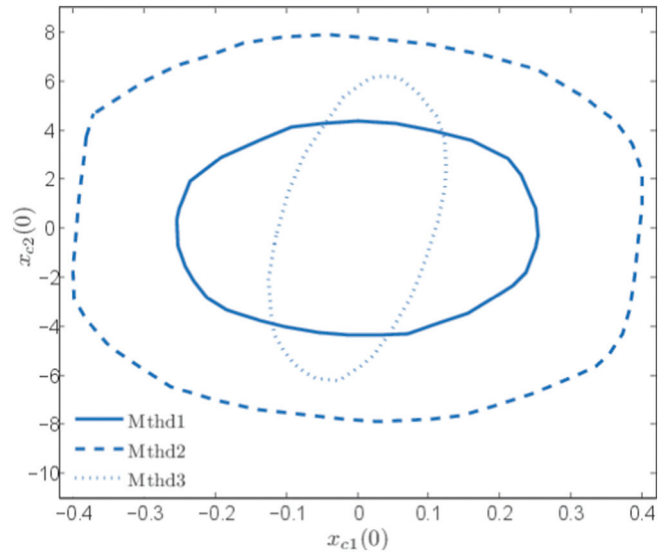


Fig. 2. The regions of attraction, set (2).

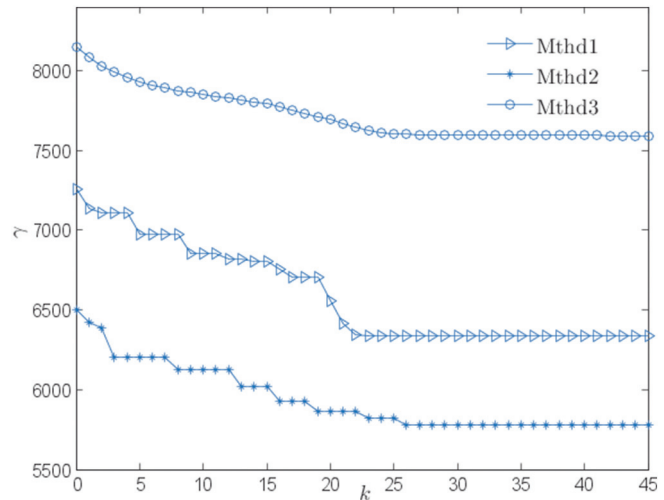


Fig. 3. The evolution of γ , set (1).

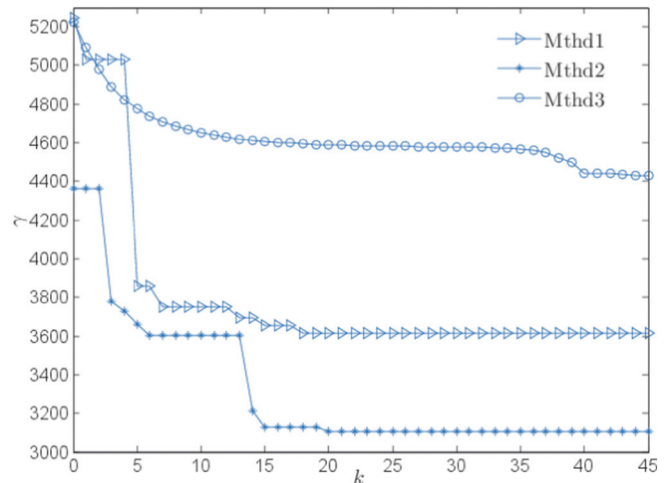


Fig. 4. The evolution of γ , set (2).

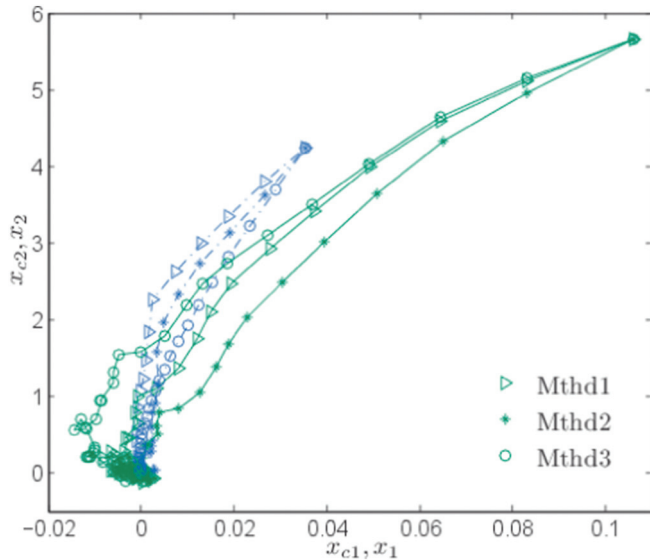


Fig. 5. The state trajectory of the closed-loop system, set (2).

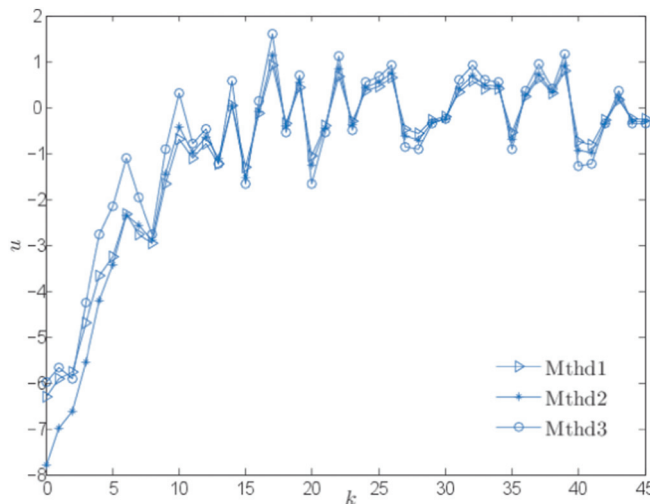


Fig. 6. The control input signal, set (2).

optimization in this paper. By observing Figs. 1–4, we obtain the following conclusions:

- (1) By comparing Mthd1 and Mthd2, it is shown that optimizing η_{rs} can enlarge the region of attraction and improve the control performance.
- (2) By comparing Mthd2 and Mthd3, it is shown that optimizing $\{F_y, L_c\}$ can enlarge the region of attraction and improve the control performance.
- (3) In Fig. 1, the region of attraction of Mthd3 even encloses that of Mthd1, while in Fig. 2 this phenomenon does not appear. The pre-specified L_c and F_y greatly influence the region of attraction and the control performance.

For Mthd1–Mthd2, since $\kappa=0.98$, i.e., 2 percent minimization error can be observed, the comparisons should be made by allowing this inaccuracy (e.g., in Figs. 3–4, the curves for Mthd1–Mthd2 decrease very

slowly or very sharply at some sampling instants). While on-line optimizing $\{F_y, L_c\}$ is advantageous for control performance and region of attraction, it is disadvantageous for computation. The total amounts of computational time for $k \leq 50$ (average of sets (1) and (2)) by applying Mthd1–Mthd3 are 0.25 h, 0.20 h and 135 s respectively. For the state trajectories and control input signals, we only give those for set II for brevity; see Figs. 5, 6, which verify the closed-loop stability and constraints satisfaction. In summary, the simulation example verifies the basic theoretical results.

6. Conclusions

This paper has presented the synthesis approaches of dynamic OFRMPC which include the previous methods in [4], [6], and [8] as special cases. The techniques for quadratic boundedness, dynamic output feedback, norm-bounding, and recursive feasibility, which were invented in separate works previously, are fused in this paper. It is shown that the fused procedure can improve the control performance and enlarge the region of attraction. We believe that this is useful, and can be a new starting point, for further researches. Based on this paper, a number of issues, such as the refreshment of EEB, the reduction of computational burden, will be studied in the future.

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